Algebra III

Semestral Examination

Instructions: $\mathbb{R}$, $\mathbb{C}$ denote the real and complex numbers respectively. All rings are assumed to be commutative with unity. All questions carry equal marks.

1. Define an integral domain. Prove that a finite integral domain is a field. Give an example of an integral domain of order 25. Justify your answer.

2. Let $C(\mathbb{R})$ denote the ring of continuous functions on $\mathbb{R}$. Define a ring homomorphism $\phi : \mathbb{R}[X, Y] \to C(\mathbb{R})$ by sending X to $\sin(t)$ and Y to $\cos(t)$. Prove that the kernel of ϕ is principal and find its generator.

3. Define a free module over a ring R. Prove that if every module over a ring R is free, then it is a field.

4. State the classification theorem for finitely generated abelian groups. Using the theorem, count the number of non-isomorphic groups of orders 100, 200 and 300.

5. Define noetherian rings. Prove that if R is a noetherian ring, then the polynomial ring $R[X]$ is noetherian.

6. Explain how a linear operator $T : V \to V$ on a vector space V over a field F makes V an $F[X]$-module. Considering the linear transformation $T(z, w) = (2z, z + w)$ on $\mathbb{C}^2$, show that it is isomorphic to $\mathbb{C}[X]/ < X^2 - 3X + 2 >$ as an $\mathbb{C}[X]$-module.